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Determining the global minimum of Higgs potentials via Groebner bases – applied to the NMSSM

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Abstract. Determining the global minimum of Higgs potentials with several Higgs fields like the next-to-minimal supersymmetric extension of the standard model (NMSSM) is a non-trivial task already at the tree level. The global minimum of a Higgs potential can be found from the set of all its stationary points defined by a multivariate polynomial system of equations. We introduce here the algebraic Groebner basis approach to solve this system of equations. We apply the method to the NMSSM with CP-conserving as well as CP-violating parameters. The results reveal an interesting stationary-point structure of the potential. Requiring the global minimum to give the electroweak symmetry breaking observed in Nature excludes large parts of the parameter space.

1 Introduction

It is a non-trivial task to find the global minimum for Higgs potentials with a large number of Higgs fields. For instance in the next-to-minimal supersymmetric extension of the standard model (NMSSM) [1–3], the Higgs sector consists of two (complex) electroweak doublets and one (complex) electroweak singlet, that is eight real fields from the doublets plus two real fields from the singlet. The conventional approach based on the unitary gauge requires the global minimum to be found in a 7-dimensional field space.

In this paper we introduce an algebraic approach to determine the global minimum of the Higgs potential. We describe how to compute all stationary points, from which then the one with the lowest value of the potential can be identified as the global minimum. We apply the method to the NMSSM, where we can reveal a quite surprising structure of stationary points; that is minima, maxima, and saddle points with different behaviour with respect to the symmetry breaking of the $SU(2)_{\mathrm{L}} \times U(1)_Y$ electroweak gauge group.

The global minimum of the Higgs potential gives the expectation values of the Higgs fields at the stable vacuum. Parameter values for the Higgs potential are thus considered acceptable only if the global minimum of the Higgs potential occurs for Higgs field vacuum expectation values which induce the spontaneous breakdown of $SU(2)_{\mathrm{L}} \times U(1)_Y$ to the electromagnetic $U(1)_{\mathrm{em}}$ at the observed electroweak scale $v \approx 246$ GeV.

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We consider the tree-level Higgs potential for general models with two Higgs doublets and an arbitrary number of additional Higgs singlets. The first step is to notice that the potential is restricted by renormalisability and gauge invariance. Renormalisability requires at most quartic terms in the real fields in the potential. Electroweak gauge invariance restricts the possible doublet terms in the potential, since only gauge invariant scalar products of doublets can occur. Substituting the doublet fields by appropriate functions of these invariant terms, we eliminate all gauge degrees of freedom from the potential and effectively reduce the occurring powers in the doublet terms. The method to base the analysis on quadratic gauge invariant functions was introduced already in the context with the general two-Higgs doublet model [4, 5].

If the potential is bounded from below, the global minima are given by the stationary points with the lowest value of the potential. The stationarity conditions form a non-linear, multivariate, inhomogeneous polynomial system of equations of third order. In this work we want to introduce a systematic approach to solve these – in general quite involved – systems of polynomial equations. We propose to determine the stationary points by a Groebner basis computation, which is well established in ideal theory [6–8]. The Groebner basis was originally introduced to solve the *ideal membership problem*. Constructing this Groebner basis in an appropriate order of the *monomials* (the terms of the polynomials including coefficients), for instance the *lexicographical ordering*, and subsequent triangularisation allows one to solve the initial system of equations algorithmically for any finite number of complex solutions. The introduction of gauge invariant functions just avoids continuous gauge symmetries in the potential and the finiteness of the set of complex solutions can eas-

ily be checked within this algorithmic approach. Moreover, this approach guarantees that all stationary points are found.

We apply the method introduced to the NMSSM. For the computation of Groebner bases as well as the subsequent steps to solve the systems of equations we employ the freely available open-source algebra program SINGULAR [9]. We find that large parts of the parameter space of the NMSSM Higgs potential can be excluded by requiring the global minimum to lead to the electroweak symmetry breaking observed in Nature. We illustrate this by determining the allowed and forbidden ranges for some generic parameter values for this model.

In the literature we found many conditions which constrain parts of the parameter space in the NMSSM; see for instance [3, 10, 11]. Typically the conditions used are necessary, but it is not clear if they are also sufficient to ensure that the resulting theory is acceptable. We also want to mention that there is a purely numerical approach to determine the global minimum in the effective one-loop NMSSM Higgs potential [12]. The aim of our present work is to systematically reveal all stationary points of the tree-level potential by solving the system of equations which originates from the stationarity condition. We show that this can be done for the full parameter space including CP-violation. With our method we can decide unequivocally if a given parameter set of the tree-level Higgs potential leads to an acceptable theory or not.

2 The method

We consider the tree-level Higgs potential of models having $SU(2)_{\rm L} \times U(1)_Y$ (weak isospin and hypercharge) electroweak gauge symmetry. In particular we study models with two Higgs doublets and n additional real Higgs isospin and hypercharge singlets. This includes in particular THDMs, where we have no additional Higgs singlets, and the NMSSM with one additional complex Higgs singlet corresponding to two real singlets. We assume both doublets to carry hypercharge $y = +1/2$ and denote the complex doublet fields by

$$\varphi_i(x) = \begin{pmatrix} \varphi_i^+(x) \\ \varphi_i^0(x) \end{pmatrix}, \quad i = 1, 2\,. \tag{1}$$

For the singlets we assume real fields which we denote by

$$\phi_i(x), \quad i = 1, \ldots, n\,. \tag{2}$$

We remark that in supersymmetric models like the NMSSM the two Higgs doublets H_d, H_u carry hypercharges $y = -1/2$ and $y = +1/2$ respectively. This can be translated to the convention used here by setting

$$\begin{aligned} \varphi_1^\alpha &= -\epsilon_{\alpha\beta}(H_d^\beta)^*\,, \\ \varphi_2^\alpha &= H_u^\alpha\,, \end{aligned} \tag{3}$$

where

$$(\epsilon_{\alpha\beta}) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{4}$$

Complex singlet fields are embedded in our notation by treating the real and imaginary parts of the complex singlets as two real singlet degrees of freedom.

2.1 Gauge invariant functions

In the most general $SU(2)_{\rm L} \times U(1)_Y$ gauge invariant Higgs potential with the field content described above, the doublet degrees of freedom enter solely via products of the following form:

$$\varphi_i^\dagger \varphi_j \quad \text{with } i, j \in \{1, 2\}\,. \tag{5}$$

It is convenient to discuss the properties of the potential such as its stability and its stationary points in terms of these gauge invariant quadratic expressions. This was discussed in detail for THDMs and also extended for the case of more than two doublets in [5]. We recall the main steps here.

We arrange all possible $SU(2)_{\rm L} \times U(1)_Y$ invariant scalar products into the hermitian 2×2 matrix

$$\underline{K} := \begin{pmatrix} \varphi_1^\dagger\varphi_1 & \varphi_2^\dagger\varphi_1 \\ \varphi_1^\dagger\varphi_2 & \varphi_2^\dagger\varphi_2 \end{pmatrix} \tag{6}$$

and consider its decomposition

$$\underline{K}_{ij} = \frac{1}{2}\left(K_0\delta_{ij} + K_a\sigma^a_{ij}\right), \tag{7}$$

where σ^a are the Pauli matrices. The four real coefficients in this decomposition are

$$K_0 = \varphi_i^\dagger\varphi_i, \quad K_a = (\varphi_i^\dagger\varphi_j)\,\sigma^a_{ij}, \quad a = 1, 2, 3, \tag{8}$$

where, here and in the following, summation over repeated indices is understood. The matrix (6) is positive semidefinite, which implies

$$K_0 \geq 0, \quad K_0^2 - K_1^2 - K_2^2 - K_3^2 \geq 0\,. \tag{9}$$

On the other hand, for every hermitian 2×2 matrix $\underline{K}_{ij}$ (7), where (9) holds there exist fields (1) satisfying (6) [5]. It was also shown in [5] that the four quantities K_0, K_a satisfying (9) parametrise the gauge orbits of the Higgs doublets. Using the inversion of (8),

$$\begin{aligned} \varphi_1^\dagger\varphi_1 &= (K_0 + K_3)/2, \quad \varphi_1^\dagger\varphi_2 = (K_1 + \mathrm{i}K_2)/2\,, \\ \varphi_2^\dagger\varphi_2 &= (K_0 - K_3)/2, \quad \varphi_2^\dagger\varphi_1 = (K_1 - \mathrm{i}K_2)/2\,, \end{aligned} \tag{10}$$

we can replace the doublet terms of the potential – due to renormalisability – by at most quadratic terms in the real functions K_0, K_1, K_2, and K_3, which simplifies the potential and eliminates all $SU(2)_{\rm L} \times U(1)_Y$ gauge degrees of freedom. We thus end up with a potential of the form $V(K_0, K_1, K_2, K_3, \phi_1, \ldots, \phi_n)$.

2.2 Stationary points

To determine the stationary points of the Higgs potential we consider $V(K_0, K_1, K_2, K_3, \phi_1, \ldots, \phi_n)$ and take the constraint (9) into account. We distinguish the possible cases of stationary points by the $SU(2)_{\mathrm{L}} \times U(1)_Y$ symmetry breaking behaviour which a vacuum of this type would have [5].

– **Unbroken** $SU(2)_{\mathbf{L}} \times U(1)_Y$: a stationary point with

$$K_0 = K_1 = K_2 = K_3 = 0. \tag{11}$$

A global minimum of this type implies a vanishing vacuum expectation value for the doublet fields (1) and therefore the trivial behaviour with respect to the gauge group. The stationary points of this type are found by setting all Higgs-doublet fields (or correspondingly the K_0 as well as the K_a fields) in the potential to zero and requiring a vanishing gradient with respect to the remaining real fields:

$$\nabla V(\phi_1, \ldots, \phi_n) = 0\,. \tag{12}$$

– **Fully broken** $SU(2)_{\mathbf{L}} \times U(1)_Y$: a stationary point with

$$\begin{aligned} K_0 &> 0, \\ K_0^2 - K_1^2 - K_2^2 - K_3^2 &> 0. \end{aligned} \tag{13}$$

A global minimum of this type has non-vanishing vacuum expectation values for the charged components of the doublets fields in (1); thus it leads to a fully broken $SU(2)_{\mathrm{L}} \times U(1)_Y$ [5]. The stationary points of this type are found by requiring a vanishing gradient with respect to all singlet fields and all gauge invariant functions:

$$\nabla V(K_0, K_1, K_2, K_3, \phi_1, \ldots, \phi_n) = 0\,. \tag{14}$$

The constraints (13) on the gauge invariant functions must be checked explicitly for the real solutions found.

– **Partially broken** $SU(2)_{\mathbf{L}} \times U(1)_Y$: a stationary point with

$$\begin{aligned} K_0 &> 0, \\ K_0^2 - K_1^2 - K_2^2 - K_3^2 &= 0. \end{aligned} \tag{15}$$

For a global minimum of this type follows the desired partial breaking of $SU(2)_{\mathrm{L}} \times U(1)_Y$ down to $U(1)_{\mathrm{em}}$ [5]. Using the Lagrange method, these stationary points are given by the real solutions of the system of equations

$$\begin{aligned} \nabla \big[V(K_0, K_1, K_2, K_3, \phi_1, \ldots, \phi_n) \\ - u(K_0^2 - K_1^2 - K_2^2 - K_3^2) \big] &= 0, \\ K_0^2 - K_1^2 - K_2^2 - K_3^2 &= 0\,, \end{aligned} \tag{16}$$

where u is a Lagrange multiplier. The inequality in (15) must be checked explicitly for the solutions found for (16).

For a potential which is bounded from below, the global minima will be among these stationary points. Solving the systems of equations (12), (14) and (16), and inserting the solutions in the potential, we can therefore identify the global minima as those solutions which have the lowest value of the potential. Note that in general there can be more than one global minimum point.

From the mathematical point of view we have with (12), (14) and (16) to solve non-linear, multivariate, inhomogeneous systems of polynomial equations of third order. We want to demonstrate that this is possible, even if the number of fields is large, like in the NMSSM. The most involved case is given by (16), which for the NMSSM consists of seven equations in seven indeterminates, namely six real fields and one Lagrange multiplier.

In the following we describe an algorithmic method to solve (12), (14) and (16) for the case that the number of complex solutions is finite. The latter is indeed fulfilled for the NMSSM with generic values for the parameters, and it is automatically checked by the method. Note that the gauge invariant functions avoid spurious continuous sets of complex solutions, which we found to arise in the case of the MSSM as well as the NMSSM if the stationarity conditions are formulated with respect to the Higgs fields (1) in a unitary gauge. This is not surprising, given the fact that the gauge invariant functions express the contribution of the doublets to the potential by four real degrees of freedom in contrast to the five encountered for the doublet components in the unitary gauge.

The solution of multivariate polynomial systems of equations is the subject of polynomial ideal theory and can be obtained algorithmically in the Groebner basis approach [6]. See Appendix A for a brief introduction to this subject. Within this approach the system of equations is transformed into a unique standard form with respect to a specified underlying ordering of the polynomial summands (*monomials*). This unique standard form of the system of equations is given by the corresponding *reduced Groebner basis*. If the underlying order is the *lexicographical* ordering, the unique standard form consists of equations with a partial separation in the indeterminates. We use a variant of the F_4 algorithm [13] to compute the Groebner bases. A Groebner basis computation is generally much faster if the standard form is computed with respect to *total degree* orderings and then transformed into a lexicographical Groebner basis. The transformation of bases from total degree to lexicographical ordering is done with the help of the FGLM algorithm [14]. Finally, the system of equations represented by the lexicographical Groebner basis has to be triangularised. The decomposition of the system of equations with a finite number of solutions into triangular sets is performed with the algorithm introduced in [15, 16]. Each triangular system consists of one univariate equation, one equation in two indeterminates, one equation in three indeterminates and so forth. This means that the solutions are found by subsequently solving just univariate equations by inserting the solutions from the previous steps.

The construction of the Groebner basis as well as the triangularisation are done algebraically, so no approxima-

tions are needed. However, the triangular system of equations contains in general polynomials of high order, where the zeros cannot be obtained algebraically. Here numerical methods are needed to find the in general complex roots of the univariate polynomials.

In more involved potentials, like the NMSSM, the algorithmic solution is considerably simplified (or even made accessible), if the coefficients of the polynomials are given in form of rational numbers. Since rational numbers are arbitrarily close to real numbers and moreover the physical parameters are given only with a certain precision this does not limit the general applicability of the method in practice.

All algorithms for the computation of the Groebner basis with respect to a given order of the monomials, the change of the underlying order, the triangularisation, and the solution of the triangular systems are implemented in the SINGULAR program package [9]. The solutions obtained can be easily checked by inserting them into the initial system of equations. Moreover, the number of complex solutions; that is, the multiplicity of the system, is known, so we can easily check that no stationary point is missing.

3 Stationary points in the NMSSM

Now, we want to apply the methods introduced in Sect. 2 to the NMSSM.

3.1 The NMSSM Higgs potential

The NMSSM Higgs sector contains two doublets and one singlet,

$$H_u = \begin{pmatrix} H_u^+ \\ H_u^0 \end{pmatrix}, \quad H_d = \begin{pmatrix} H_d^0 \\ H_d^- \end{pmatrix}, \quad S, \tag{17}$$

with the tree-level Higgs potential $V_{\mathrm{NMSSM}} = V_{\mathrm{F}} + V_{\mathrm{D}} + V_{\mathrm{soft}}$ [3, 11], where

$$\begin{aligned} V_{\mathrm{F}} &= |\lambda S|^2 \left(|H_u|^2 + |H_d|^2\right) + \left|\lambda H_u H_d + \kappa S^2\right|^2 , \\ V_{\mathrm{D}} &= \frac{1}{8}\bar{g}^2 \left(|H_d|^2 - |H_u|^2\right)^2 + \frac{1}{2} g^2 \left|H_u^\dagger H_d\right|^2 , \\ V_{\mathrm{soft}} &= m_{H_u}^2 |H_u|^2 + m_{H_d}^2 |H_d|^2 + m_S^2 |S|^2 \\ &\quad + \left[\lambda A_\lambda S H_u H_d + \frac{1}{3}\kappa A_\kappa S^3 + \mathrm{h.c.}\right] . \end{aligned} \tag{18}$$

We use the notation $H_u H_d \equiv \epsilon_{\alpha\beta} (H_u)^\alpha (H_d)^\beta = H_u^+ H_d^- - H_u^0 H_d^0$ and $\bar{g} = \sqrt{g^2 + g'^2}$, where g and g' are the $SU(2)_{\mathrm{L}}$ and $U(1)_Y$ gauge couplings, respectively. The parameters of the potential are given by the experimentally fixed electroweak gauge couplings and

$$\lambda, \kappa, m_{H_u}^2, m_{H_d}^2, m_S^2, A_\lambda, A_\kappa. \tag{19}$$

The quartic terms of the potential (18) are positive for any non-trivial field configuration, if both λ and κ are non-vanishing. The potential is therefore bounded from below for all cases considered here, and stability need not to be checked any further.

We translate the NMSSM Higgs potential to the formalism described in the previous section, where we decompose the complex singlet field into two real fields according to $S = S_{\mathrm{re}} + \mathrm{i} S_{\mathrm{im}}$. In this notation the potential is given by

$$\begin{aligned} V_{\mathrm{F}} &= \frac{1}{4}|\lambda|^2 \left(K_1^2 + K_2^2 + 4K_0 \left(S_{\mathrm{re}}^2 + S_{\mathrm{im}}^2\right)\right) \\ &\quad + |\kappa|^2 \left(S_{\mathrm{re}}^2 + S_{\mathrm{im}}^2\right)^2 \\ &\quad - \mathrm{Re}(\lambda\kappa^*) \left(K_1 \left(S_{\mathrm{re}}^2 - S_{\mathrm{im}}^2\right) + 2K_2 S_{\mathrm{re}} S_{\mathrm{im}}\right) \\ &\quad + \mathrm{Im}(\lambda\kappa^*) \left(K_2 \left(S_{\mathrm{re}}^2 - S_{\mathrm{im}}^2\right) - 2K_1 S_{\mathrm{re}} S_{\mathrm{im}}\right) , \\ V_{\mathrm{D}} &= \frac{1}{8}\bar{g}^2 K_3^2 + \frac{1}{8} g^2 \left(K_0^2 - K_1^2 - K_2^2 - K_3^2\right) , \\ V_{\mathrm{soft}} &= \frac{1}{2} m_{H_u}^2 \left(K_0 - K_3\right) + \frac{1}{2} m_{H_d}^2 \left(K_0 + K_3\right) \\ &\quad + m_S^2 \left(S_{\mathrm{re}}^2 + S_{\mathrm{im}}^2\right) \\ &\quad - \mathrm{Re}(\lambda A_\lambda) \left(K_1 S_{\mathrm{re}} - K_2 S_{\mathrm{im}}\right) \\ &\quad + \mathrm{Im}(\lambda A_\lambda) \left(K_2 S_{\mathrm{re}} + K_1 S_{\mathrm{im}}\right) \\ &\quad + \frac{2}{3} \mathrm{Re}(\kappa A_\kappa) \left(S_{\mathrm{re}}^3 - 3 S_{\mathrm{re}} S_{\mathrm{im}}^2\right) \\ &\quad + \frac{2}{3} \mathrm{Im}(\kappa A_\kappa) \left(S_{\mathrm{im}}^3 - 3 S_{\mathrm{re}}^2 S_{\mathrm{im}}\right) . \end{aligned} \tag{20}$$

For given values of the potential parameters (19) we can find all stationary points of the NMSSM by solving the systems of equations (12), (14) and (16) as described above.

3.2 Choice of parameters

In order to fix experimentally known parameters like the electroweak scale it is inappropriate to choose numerical values for the parameter set (19). Instead, we express different original parameters in terms of the desired vacuum expectation values of the neutral components of the Higgs doublets and the Higgs singlet, the mass of the charged Higgs boson and a CP-violating phase. To this end we introduce the vacuum expectation values of the Higgs fields,

$$\begin{aligned} \langle H_u \rangle &= \mathrm{e}^{\mathrm{i}\varphi_u} \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} v_u \end{pmatrix} , \\ \langle H_d \rangle &= \begin{pmatrix} \frac{1}{\sqrt{2}} v_d \\ 0 \end{pmatrix} , \\ \langle S \rangle &= \frac{1}{\sqrt{2}} \mathrm{e}^{\mathrm{i}\varphi_S} v_S , \end{aligned} \tag{21}$$

which are parametrised by v_u, v_d, v_S, and the phases φ_u, and φ_S. The stationarity condition of the potential at the vacuum values for the fields requires a vanishing gradient with respect to the Higgs fields. These equations relate the parameters of the quartic terms as well as the soft-breaking mass parameters with the vacuum expectation values. As usual, we define $v^2 \equiv v_u^2 + v_d^2$ and $\tan\beta \equiv v_u / v_d$. Further, if we write the complex parameters λ, κ, A_λ, and A_κ in polar

coordinates with phases δ_λ, δ_κ, δ_{A_λ}, δ_{A_κ} and introduce the abbreviations

$$\delta_{\mathrm{EDM}} \equiv \delta_\lambda + \varphi_u + \varphi_S, \quad \delta'_\kappa \equiv \delta_\kappa + 3\varphi_S\,, \tag{22}$$

the initial parameters of the potential (19) can be replaced by the new set of parameters

$$\lambda, \kappa, |A_\kappa|, \tan\beta, v_S, m_{H^\pm}, \mathrm{sign}\, R_\kappa, \delta_{\mathrm{EDM}}, \delta'_\kappa\,, \tag{23}$$

plus the electroweak scale $v \approx 246\,\mathrm{GeV}$. Note that it is not sufficient to supply the length $|A_\kappa|$; in addition we have to fix the sign of

$$R_\kappa \equiv \frac{1}{\sqrt{2}}\,\mathrm{Re}\left(\kappa A_\kappa \mathrm{e}^{\mathrm{i}3\varphi_S}\right)\,. \tag{24}$$

In the mass matrix of the Higgs scalars, the CP-violating entries which mix the "scalar" with the "pseudoscalar" fields are proportional to the imaginary part of $\exp[\mathrm{i}(\delta_{\mathrm{EDM}} - \delta'_\kappa)]$.

3.3 Numerical results

As a numerical example we choose the parameter values

$$\begin{aligned} \lambda = 0.4, \quad \kappa = 0.3, \quad |A_\kappa| = 200\,\mathrm{GeV}, \\ \tan\beta = 3, \quad v_S = 3v, \quad m_{\mathrm{H}^\pm} = 2v, \\ \mathrm{sign} R_\kappa = -, \quad \delta_{\mathrm{EDM}} = 0, \quad \delta'_\kappa = 0 \end{aligned} \tag{25}$$

and consider the variation of one parameter at a time with the values of the other parameters in (25) kept fixed. For a given point in parameter space we compute all stationary points of the NMSSM potential as described above.

As mentioned in Sect. 2.2 the Groebner basis construction is performed with numerical coefficients. Here we use a precision of 12 digits for the input parameters (19), which are determined from the values for the parameters (23). The roots of the univariate polynomials are found numerically, where we choose a precision of 100 digits. Our approach allows one to use arbitrary precisions in both cases. We verify that the errors of the approximate statements described in the following are under control.

For generic values of the parameters we find 52 complex solutions: seven corresponding to the unbroken, 38 to the partially broken, and seven to the fully broken cases. The number of real and therefore relevant solutions depends on the specific values of the parameters.

As expected from the Z_3 symmetry of the potential, we find either one or three solutions sharing the same value of the potential within the accuracy of the numerical roots. From the computed stationary points only those may be accepted as global minima which correspond to the initial vacuum expectation values (up to the complex phases); that is, which fulfill

$$\sqrt{2K_0} \approx v, \quad \sqrt{\frac{K_0 - K_3}{K_0 + K_3}} \approx \tan\beta, \quad \sqrt{2\left(S_{\mathrm{re}}^2 + S_{\mathrm{im}}^2\right)} \approx v_S\,. \tag{26}$$

Since for non-vanishing λ, κ the potential is bounded from below, the stationary point with the lowest value of the potential is the global minimum.

Further, we determine for every stationary solution, whether it is a local maximum, local minimum or a saddle point. For the regular solutions, i.e. the partially and fully broken cases, this is achieved via the bordered Hessian matrix (see for instance [17]) in terms of $K_0, K_1, K_2, K_3, S_{\mathrm{re}}, S_{\mathrm{im}}$. This takes all powers of the doublet fields into account, which allows for a definite decision on the type of the stationary point also for frequently encountered partial breaking solutions where at least one mass squared is zero and the others have the same sign. For irregular solutions, i.e. the non-breaking solutions with $K_0 = 0$, the Lagrange formalism cannot be used since the gradients of the two boundary conditions with respect to $K_0, K_1, K_2, K_3, S_{\mathrm{re}}, S_{\mathrm{im}}$ become linearly dependent. Instead we resubstitute the original fields $H_u^+, H_u^0, H_d^0, H_d^-, S_{\mathrm{re}}, S_{\mathrm{im}}$ in this case and consider the free Hessian matrix with respect to these fields. This turns out to be sufficient in practice to judge on the type of the stationary points.

Figures 1 and 2 show the values of the potential at all stationary points for the parameter values (25) and the cases where successively one of the parameters $\kappa, \lambda, \tan\beta, v_S, m_{\mathrm{H}^\pm}, \delta'_\kappa$ is varied.

Each curve in the figures represents 1- or 3-fold degenerate stationary potential values, where the gauge symmetry breaking behaviour of the solutions is denoted by different line styles. Excluded parameter regions, where the global minimum does not exhibit the required expectation values (26), are drawn shaded. As is illustrated by the figures we find that substantial regions of the NMSSM parameter space are excluded. For some excluded parameter regions, the partially breaking solutions with the required vacuum expectation values (26) are saddle points (see for instance Fig. 1, top). This means they can be discarded as global minima without calculation of the other stationary points. However, this is not always the case. Obviously from Fig. 2, top, we find an upper bound for v_S. For the plotted v_S larger than this upper exclusion bound the solutions fulfilling (26) are still pronounced minima, i.e. the mass matrices have positive eigenvalues, but they are no longer the global minima. The influence of the CP-violating phase δ'_κ is shown in Fig. 2, bottom. Note that $\delta'_\kappa \to -\delta'_\kappa$ is not a symmetry of the potential. However, the potential is invariant under $(\delta'_\kappa, K_2, S_{\mathrm{im}}) \to -(\delta'_\kappa, K_2, S_{\mathrm{im}})$; that is, $(\delta'_\kappa, H_u, H_d, S) \to (-\delta'_\kappa, H_u^*, H_d^*, S^*)$, if the residual phases are chosen as in (25). Therefore the stationary values of the potential in Fig. 2 depend only on $|\delta'_\kappa|$. Also we want to note that in all figures shown here there are non-breaking saddle points with potential values slightly above those of the *wanted* global minimum. We find that this effect is not coincidental for the parameters (25) chosen here, but rather a generic feature of the NMSSM. Within the CP-conserving parameter range

$$\begin{aligned} \lambda \in]0,1], \quad \kappa \in]0,1], \quad A_\kappa \in \pm]0,2500]\mathrm{GeV}, \quad \tan\beta \in]0,50], \\ v_s \in]0,5000]\mathrm{GeV}, \quad m_{\mathrm{H}^\pm} \in]0,2500]\mathrm{GeV}\,, \end{aligned} \tag{27}$$

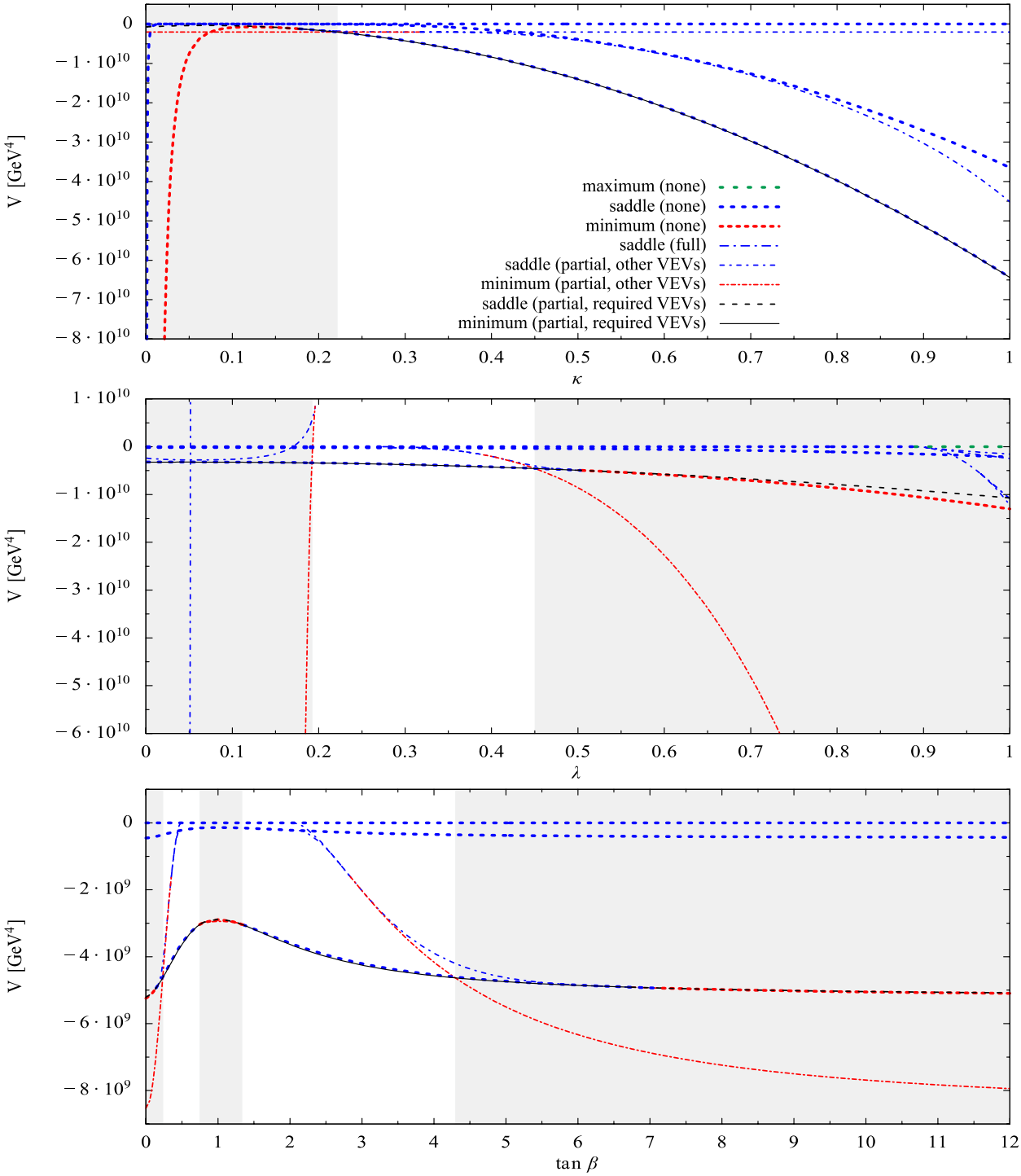

Fig. 1. Values of the NMSSM potential at its stationary points in dependence of $\kappa, \lambda, \tan\beta$. The following parameters are kept constant unless explicitly varied: $\lambda = 0.4, \kappa = 0.3, |A_\kappa| = 200\,\text{GeV}, \tan\beta = 3, v_S = 3v, m_{H^\pm} = 2v, \text{sign}\, R_\kappa = -, \delta_{\text{EDM}} = \delta'_\kappa = 0$. Each line corresponds to 1 or 3 stationary points sharing the same value of the potential. The different line styles denote saddle points, maxima, and minima. The labels 'none', 'full', and 'partial' denote solutions of the classes with unbroken (12), fully broken (14), and partially broken (16) $SU(2)_\text{L} \times U(1)_Y$, respectively. For solutions of the partially broken class, it is also denoted whether they correspond to the 'required VEVs' v_u, v_d, v_S or to 'other VEVs'. Excluded parameter values, where the global minimum does not exhibit the required vacuum expectation values, are drawn *shaded*

we select samples producing the wanted global minimum and typically find non-breaking saddle points, where the relative separation of the potential values for the saddle points and the global minimum is below the per-mille level, in many cases even far below. We do not find fully breaking global minima for scenarios in the

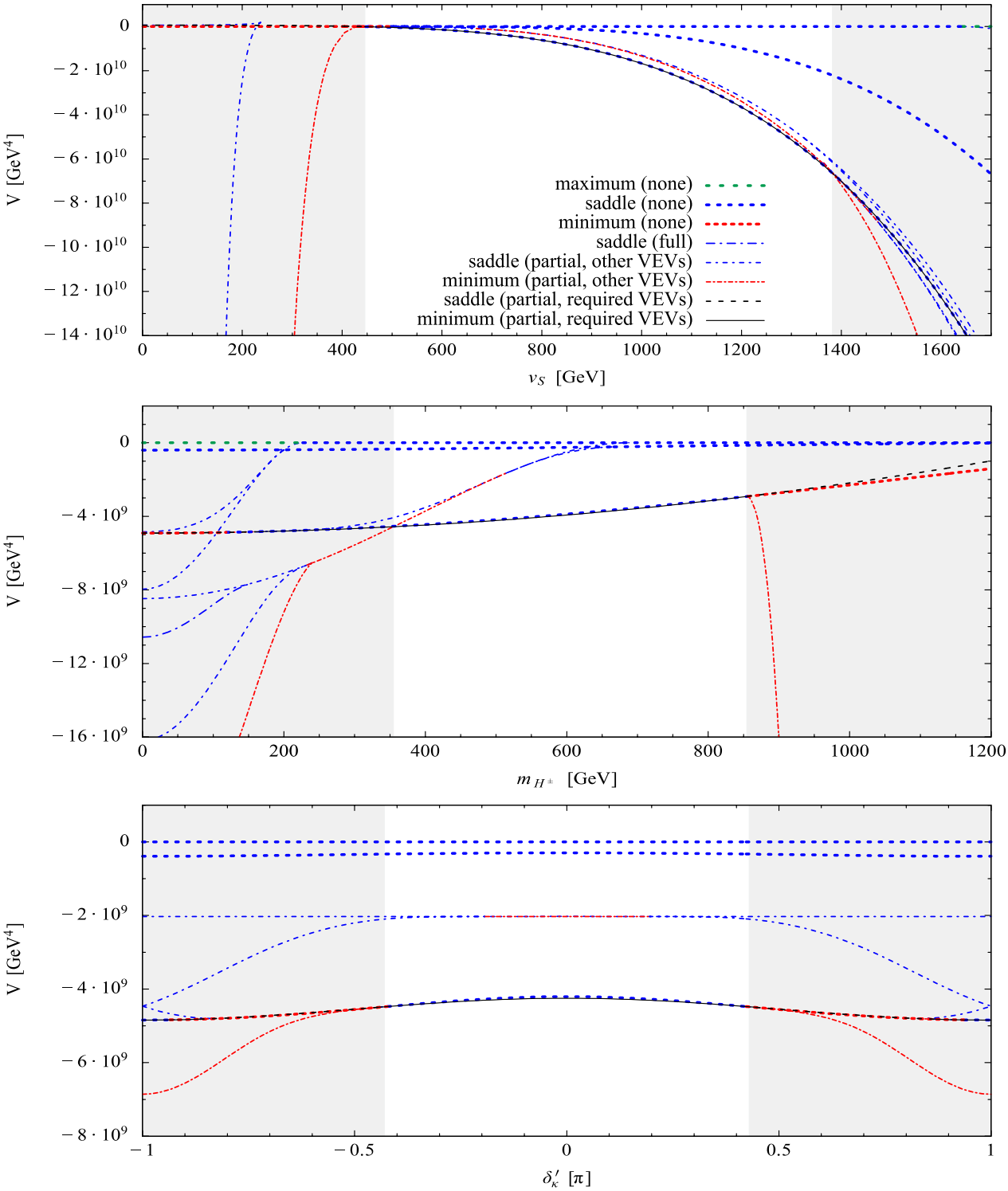

Fig. 2. Same as in Fig. 1 but for variation of $v_S, m_{H^\pm}, \delta'_\kappa$, respectively

range (27) where the solutions with the required vacuum expectation values (26) are local minima. Eventually, we find examples, where CP-conserving parameter values with the "wrong" global minimum produce the wanted global minimum if a non-vanishing phase δ'_κ is introduced.

4 Conclusion

The stationarity conditions of Higgs potentials at the tree level lead in general to non-linear, multivariate, polynomial systems of equations. We have formulated the stationarity conditions by means of gauge invariant functions,

thus avoiding to have to deal with electroweak gauge degrees of freedom. We have introduced the Groebner basis approach to solve the systems of equations and have applied the method to the NMSSM. Within the Groebner basis approach we have easily found all stationary solutions even with general CP-violating parameters. We have demonstrated that large regions of the parameter space of the NMSSM are excluded because they do not lead to the required global minimum. Finally, we note that the method proposed here should also be applicable to other potentials which neither have to be restricted to two doublets nor even have to be renormalisable.

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Appendix : Buchberger algorithm

In this appendix we want to sketch the construction of the Buchberger algorithm which transforms a given set of polynomials F into a Groebner basis G. The Groebner basis G has exactly the same simultaneous zeros as the initial set of polynomials F, but allows for better access to the actual calculation of these zeros.

The general idea is to *complete* the set F by adjoining differences of polynomials. Before we can present the algorithms themselves we have to introduce the two basic ingredients; that is, *reduction* and the *S-polynomial*. For a more detailed discussion we refer the reader to the literature [6–8]. Here we follow closely [8]. First of all we recall some definitions.

Definition 1. *Polynomial ring*
A polynomial ring $K[x_1,\dots,x_n] \equiv K[\mathbf{x}]$ is the set of all n-variate polynomials with variables $x_1,\dots,x_n$ and coefficients in the field K.

Definition 2. *Generated ideal*
Let $F = \{f_1,\dots,f_n\} \subset K[\mathbf{x}]$ be finite; F generates an ideal defined by

$$I(F) \equiv \left\{ \sum_{f_i \in F} r_i f_i \middle| r_i \in K[\mathbf{x}], f_i \in F, i = 1,\dots,n \right\}.$$

In the following we want to consider an explicit example; that is, a set $F = \{f_1, f_2, f_3\} \subset \mathbb{Q}[x,y]$ of polynomials with rational coefficients:

$$\begin{aligned} f_1 &= 3x^2y + 2xy + y + 9x^2 + 5x - 3\,, \\ f_2 &= 2x^3y - xy - y + 6x^3 - 2x^2 - 3x + 3\,, \\ f_3 &= x^3y + x^2y + 3x^3 + 2x^2\,. \end{aligned} \tag{A.1}$$

The set F generates an ideal $I(F)$, which is given by the set of sums of f_1, f_2, and f_3, where each polynomial is multiplied with another arbitrary polynomial from the ring $\mathbb{Q}[x,y]$. The summands of the polynomial are denoted as *monomials* and each monomial is the product of a coefficient and a *power product*.

Further, we introduce an *ordering* ($\succ$) of the monomials. In the *lexicographical ordering* ($\succ_{\text{lex}}$) the monomials are ordered with respect to the power of each variable subsequently. The ring notation $\mathbb{Q}[x,y]$ defines $y \succ_{\text{lex}} x$; that is, for the lexicographical ordering of monomials, powers of y are considered first, then powers of x. Explicitly, this means $2x^2y^3 \succ_{\text{lex}} 5xy^2$, because the power of y is larger in the first monomial and $2xy^2 \succ_{\text{lex}} 5y^2$, because both monomials have the same power of y, but the first monomial has a larger power of x. The monomials of the polynomials (A.1) from the ring $\mathbb{Q}[x,y]$ are ordered with respect to lexicographical ordering. In *total degree ordering* ($\succ_{\text{deg}}$) the monomials are ordered with respect to the sum of powers in each monomial. If two monomials have the same sum of powers, they are ordered with respect to another ordering, for instance lexicographical. For polynomials in $\mathbb{Q}[x,y]$ we have $x^2y \succ_{\text{deg}} 4xy$, since the sum of powers of the left power product is 3 compared to 2 for the right power product.

The largest power product with respect to the underlying ordering ($\succ$) of a polynomial f is denoted as the *leading power product*, $\text{LP}(f)$, the corresponding coefficient as *leading coefficient*, $\text{LC}(f)$. With the help of these preparations we can define the two essential parts of the Buchberger algorithm; that is, *reduction* and the *S-polynomial*.

Definition 3. *Reduction*
Let $f, p \in K[\mathbf{x}]$. We call f reducible modulo p, if for a power product t of f there exists a power product u with $\text{LP}(p)u = t$. Then, we say, f reduces to h modulo p, where $h = f - \frac{Coefficient(f,t)}{\text{LC}(p)} up$.

In the example (A.1) the polynomial f_3 is reducible modulo f_1, since for example the second monomial of f_3, that is x^2y, is a multiple of the $\text{LP}(f_1)$, and $h = f_3 - 1/3 f_1 = x^3y - 2/3xy - 1/3y + 3x^3 - x^2 - 5/3x + 1$.

Reduction of a polynomial modulo a set $P \subset K[\mathbf{x}]$ is accordingly defined if there is a $p \in P$ such that f is reducible modulo p. Further, we say that a polynomial h is in *reduced form* or *normal form* modulo F, in short $\text{normf}(h,F)$, if there is no h' such that h reduces to h' modulo F. A set $P \subset K[\mathbf{x}]$ is called reduced, if each $p \in P$ is in reduced form modulo $P \backslash \{p\}$. Note that reduction is defined with respect to the underlying ordering of the monomials, since the leading power product is defined with respect to the ordering. In general, a normal form is not unique, neither for a polynomial nor for a set.

Now we can present an algorithm to compute a normal form $Q \subset K[\mathbf{x}]$ of a finite $F \subset K[\mathbf{x}]$.

Algorithm. *Normal form*
For a given finite set $F \subset K[\mathbf{x}]$ determine a normal form $Q \subset K[\mathbf{x}]$:

$Q := F$
while *exists* $p \in Q$
which is reducible modulo $Q\backslash\{p\}$ **do**
  $Q := Q\backslash\{p\}$
  $h := \mathrm{normf}(p, Q)$
  if $h \neq 0$ **then**
    $Q := Q \cup \{h\}$
return Q

Clearly, the simultaneous zeros of all $f_i \in F$ are also simultaneous zeros of all $q_i \in Q$ and vice versa.

Definition 4. *S-polynomial*
For $g_1, g_2 \in K[\mathbf{x}]$ *the S-polynomial of* g_1 *and* g_2 *is defined as*

$$\mathrm{spol}(g_1, g_2) \equiv \frac{\mathrm{lcm}\left(\mathrm{LP}(g_1), \mathrm{LP}(g_2)\right)}{\mathrm{LP}(g_1)} g_1 - \frac{\mathrm{LC}(g_1)}{\mathrm{LC}(g_2)} \frac{\mathrm{lcm}\left(\mathrm{LP}(g_1), \mathrm{LP}(g_2)\right)}{\mathrm{LP}(g_2)} g_2 \,,$$

where lcm *denotes the least common multiple.*

Clearly, a simultaneous zero of g_1 and g_2 is also a zero of $\mathrm{spol}(g_1, g_2)$. In the example (A.1) we can build the S-polynomial for any two polynomials, for instance

$$\begin{aligned}\mathrm{spol}(f_1, f_2) &= \frac{x^3 y}{x^2 y}\, f_1 - \frac{3}{2}\, \frac{x^3 y}{x^3 y}\, f_2 = x\, f_1 - 3/2\, f_2 \\ &= 2x^2 y + 5/2 xy + 3/2 y + 8x^2 + 3/2 x - 9/2 \,.\end{aligned}$$

Finally we define the Groebner basis.

Definition 5. *Groebner basis*
$G \subset K[\mathbf{x}]$ *is called a Groebner basis, if for all* $f_1, f_2 \in G$ $\mathrm{normf}(\mathrm{spol}(f_1, f_2), G) = 0$.

Now we are in a position to present the Buchberger algorithm.

Algorithm. *Buchberger*
For a given finite set $F \subset K[\mathbf{x}]$ *determine the Groebner basis* $G \subset K[\mathbf{x}]$ *with* $I(F) = I(G)$*:*

$G := F$
$B := \{\{g_1, g_2\} | g_1, g_2 \in G \text{ with } g_1 \neq g_2\}$
while $B \neq \emptyset$ **do**
  choose $\{g_1, g_2\}$ *from* B
  $B := B\backslash\{\{g_1, g_2\}\}$
  $h := \mathrm{spol}(g_1, g_2)$
  $h' := \mathrm{normf}(h, G)$
  if $h' \neq 0$ **then**
    $B := B \cup \{\{g, h'\} | g \in G\}$
    $G := G \cup \{h'\}$
return G

Note that since G just follows by adjoining reduced S-polynomials to F both sets generate the same ideal; especially, both sets have exactly the same simultaneous zeros.

It can be proven that the Buchberger algorithm terminates.

The final step is to construct the reduced Groebner basis by applying the normal form algorithm defined above to the Groebner basis G. It can be shown that the reduced Groebner basis is unique [7]. If we apply the Buchberger algorithm to the set (A.1) with subsequent reduction we end up with the reduced Groebner basis (with underlying lexicographical ordering):

$$\begin{aligned} g_1 &= y + x^2 - 3/2 x - 3 \,, \\ g_2 &= x^3 - 5/2 x^2 - 5/2 x \,. \end{aligned}$$

The system of equations $g_1 = g_2 = 0$ is equivalent to $f_1 = f_2 = f_3 = 0$, but the former allows one to directly calculate the solutions: since $g_2 = 0$ is univariate it can be solved immediately and subsequently $g_1 = 0$ for each partial solution inserted.

Despite the correctness of the Buchberger algorithm tractability of practical examples requires one to improve this algorithm. In particular, the number of iterations in the algorithm drastically grows with increasing number of polynomials and with higher degrees of the polynomials. In this respect much progress has been made with the improvement of this original Buchberger algorithm from 1965 [7, 8, 13].

References

1. P. Fayet, Nucl. Phys. B **90**, 104 (1975)
2. M. Drees, Int. J. Mod. Phys. A **4**, 3635 (1989)
3. J.R. Ellis, J.F. Gunion, H.E. Haber, L. Roszkowski, F. Zwirner, Phys. Rev. D **39**, 844 (1989) and other references quoted therein
4. F. Nagel, New aspects of gauge-boson couplings and the Higgs sector, PhD-thesis, Heidelberg University (2004)
5. M. Maniatis, A. von Manteuffel, O. Nachtmann, F. Nagel, hep-ph/0605184, to be published in Eur. Phys. J. C
6. B. Buchberger, Ein Algorithmus zum Auffinden der Basiselemente des Restklassenringes nach einem nulldimensionalen Polynomideal, PhD-thesis, University Innsbruck (1965)
7. T. Becker, V. Weispfenning, Gröbner Bases (Springer, New York, 1993)
8. N.K. Bose, J.P. Guiver, E.W. Kamen, H.M. Valenzuela, B. Buchberger, D. Reidel, Multidimensional Systems Theory, Progress, Directions and Open Problems in Multidimensional Systems (Publishing Company, 1985)
9. G.M. Greuel, G. Pfister, H. Schoenemann, SINGULAR – A Computer Algebra System for Polynomial Computations, in: Symbolic Computation and Automated Reasoning, ed. by M. Kerber, M. Kohlhase, The Calculemus-2000 Symposium (2001) 227; SINGULAR is available at http://www.singular.uni-kl.de
10. U. Ellwanger, J.F. Gunion, C. Hugonie, JHEP **0502**, 066 (2005) [hep-ph/0406215]
11. D.J. Miller, R. Nevzorov, P.M. Zerwas, Nucl. Phys. B **681**, 3 (2004) [hep-ph/0304049]
12. K. Funakubo, S. Tao, Prog. Theor. Phys. **113**, 821 (2005) [hep-ph/0409294]

13. J.-C. Faugère, J. Pure. Appl. Algebra **139**, 61 (1999)
14. J.-C. Faugère, P. Gianni, D. Lazard, T. Mora, J. Symb. Comput. **16**, 329 (1993)
15. H.M. Moeller, Appl. Algebr. Eng. Commun. **4**, 217 (1993)
16. D. Hillebrand, Triangulierung nulldimensionaler Ideale – Implementierung und Vergleich zweier Algorithmen, PhD-thesis, University Dortmund (1999)
17. R. Fletcher, Practical Methods of Optimization, 2nd Ed. (Wiley, Chichester, New York, 1987)